CONTROL OF NON-AXISYMMETRIC MAGNETIC FIELDS IN TOKAMAKS

error fields, resistive wall modes, beneficial perturbations

Allen Boozer
Columbia University

I. Non-axisymmetric magnetic perturbations
II. Constraints of mathematics and Maxwell's equations
III. Required knowledge
IV. Error field control
V. Resistive Wall Modes
VI. Beneficial Perturbations

Talk is a summary of a manuscript
Control of non-axisymmetric magnetic field perturbations in tokamaks
For a copy email me: ahb17@columbia.edu

NON-AXISYMMETRIC MAGNETIC PERTURBATIONS

1. Magnetic field errors

   Perturbations as small as $\delta B/B \sim 10^{-4}$ can cause disruptions. Places great demands on construction accuracy without control coils.

2. Resistive wall mode

   An ideal MHD instability that prevents tokamaks having an adequate bootstrap current for steady state operation.

   Would be stabilized if the surrounding structures were perfectly conducting and grows on their resistive time scale unless stabilized by feedback using control coils.

3. Beneficial Perturbations

   Best-known example is ELM avoidance using control coils.

CONSTRAINTS OF MATHEMATICS & MAXWELL'S EQS.

A common—but false—implicit assumption:

Integrated plasma computations are superior to a separation of information that depends on the plasma model from the constraints of mathematics and Maxwell's equations.

Detailed plasma models should provide only what is not provided by mathematics and Maxwell's equations:

The spectrum of external normal magnetic field distributions on the plasma boundary to which the plasma is sensitive and the required amplitudes for significant effects.

Integrated plasma computations can be sensitive to the plasma model and its physical consistency.

REQUIRED KNOWLEDGE

Required for a sensible discussion of control.

1. Strong poloidal coupling
2. Spectrum of plasma sensitivities
3. Limits on independent control
4. Importance of uncontrolled distributions
5. Limits on control coils
6. Important machine errors
7. Two-fold degeneracy of distributions
8. Plasma response (global and local)

Some Paradoxes Resolved

1. Successful error field control does not mean error field reduction.

The error field control system on DIII-D <u>*increases*</u> *the toroidal asymmetry of the magnetic field when it optimally mitigates the effect of the error field.*

2. The optimal location for error field control coils may be rotated from the poloidal location of the source of the error.

On NSTX an inboard field error is controlled by an outboard magnetic field about twenty times smaller.

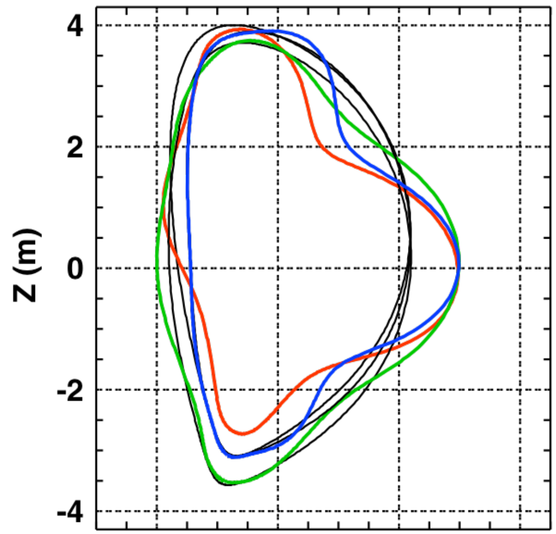
3. The drive for islands at the *q=2* surface has little to do with the resonant part *m=2,n=1* part of the external perturbation.

The Fourier component of the external magnetic field with the largest drive for islands at the q=2 surface is m~10,n=1.

1. Strong Poloidal Coupling

The dependence of the tokamak geometry on the poloidal angle gives a strong coupling between different poloidal Fourier modes.

$\delta\vec{B}_n \cdot \hat{n}$ to which ITER q=2 Surface Most Sensitive



Black contours give plasma boundary

Deviations of colored contours give magnitude of $\delta\vec{B}_n \cdot \hat{n}$ with *N=1.*

Park et al, Nucl. Fusion **48**, 045006 (2008)

ITER error field control system was designed to control *n=1, m=1,2,3* external perturbations to eliminate drive for *q=2* island.

2. Spectrum of Plasma Sensitivities

A tokamak plasma has a broad range of sensitivities to different distributions of the external magnetic field.

$\delta\vec{B}_x$ due to currents outside the plasma is completely characterized in the plasma volume by $\delta\vec{B}_x \cdot \hat{n}$ on the plasma boundary.

Two external magnetic distributions are orthogonal if $\oint \left(\delta\vec{B}_x \cdot \hat{n}\right)_i \left(\delta\vec{B}_x \cdot \hat{n}\right)_j \frac{da}{w} = 0$, where $\oint w(\theta)da = 1$ and $w(\theta) > 0$.

The minimum amplitude for a magnetic distribution to have significant effects defines the magnetic flux $\sigma_i \equiv \sqrt{\oint \left(\delta\vec{B}_x \cdot \hat{n}\right)_i^2 \frac{da}{w}}$.

The distribution to which the plasma is most sensitive has the smallest σ_i.

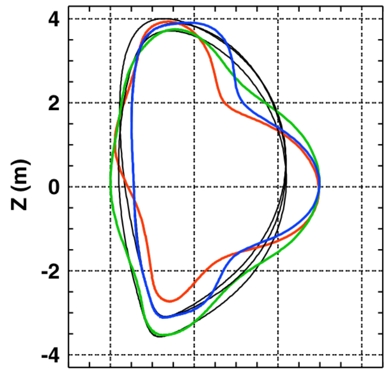
Calculations Limited by Missing Plasma Information

Need spectrum of external magnetic field distributions.

$\left(\delta\vec{B}_x \cdot \hat{n}\right)_1$ is the perturbation with the smallest σ.

$\left(\delta\vec{B}_x \cdot \hat{n}\right)_2$ is the orthogonal perturbation with the next smallest σ, etc.

External distribution to which the plasma is most sensitive is known.
- Drives what would be the resistive wall mode.
- Greatly amplified by displacing plasma equilibrium currents.
- Gives large non-axisymmetric effect throughout plasma.



Little known empirically or theoretically about distributions of secondary sensitivity.

3. Limits on Independent Control

The difficulty of independently controlling magnetic distributions increases exponentially with their number.

a. If magnetic distributions are ordered by the current required to drive them, the required current increases exponentially.

b. If one distribution requires exponentially more current than another, the coil currents for the hard to drive distribution must be specified with exponentially great accuracy to avoid also producing the distribution that requires little current.

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However if the source of magnetic field errors is further from the plasma than the control coils,

Control of arbitrary field errors is feasible in principle.

For ITER: True for inaccuracies in coils.
False for inaccuracies in ferritic inserts.

4. Importance of Uncontrolled Distributions

Only magnetic distributions to which the plasma is sensitive can usefully be controlled.

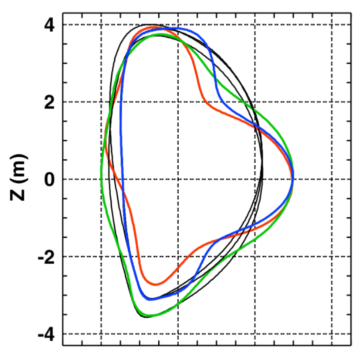
All other magnetic distributions should be left uncontrolled to have a practical control system—avoid exponential increase in difficulty.

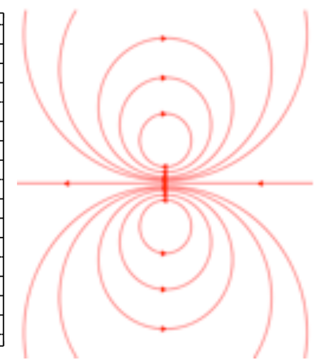
5. Limits on Control Coils

Coils well separated from the plasma cannot accurately produce magnetic distributions of practical importance.

The $\left(\delta\vec{B}_x \cdot \hat{n}\right)_i$ to which the plasma is most sensitive has a single-sign lobe of width Δ on the outboard side. For ITER *1/2 > Δ /a >1/3*.

A small coil looks like a magnetic dipole. The radius of the region of single-sign field is $\sqrt{2}x$ with *x* the separation. Need $\frac{x}{a} \leq \frac{\Delta/a}{2\sqrt{2}} \sim \frac{1}{8}$.





External field with poloidal mode number m, $\delta\vec{B}_x \cdot \hat{n} \propto e^{-m\ln(b/a)} I_b$,
b is coil and *a* is plasma radius. High *m's* are exponentially difficult.

Normal Magnetic Field to Plasma Boundary gives Fluxes

$\delta\vec{B}\cdot\hat{n} = w\sum_i \Phi_i f_i(\theta,\varphi)$, where $\oint f_i f_j w(\theta)da = \delta_{ij}$ with $\oint w(\theta)da = 1$.

The $f_i(\theta,\varphi)$ are expansion (such as Fourier) functions; $\Phi_i = \oint f_i\delta\vec{B}\cdot d\vec{a}$

Magnetic flux components Φ_i form a matrix column vector $\vec{\Phi}$.

Surface Current on Plasma Boundary gives Currents

Currents on a surface have form $\vec{j}_s = \vec{\nabla}\times\left(\kappa(\theta,\varphi)\delta(r-a)\vec{\nabla}r\right)$.

Current potential has units of Amperes $\kappa(\theta,\varphi) = \sum_j J_j f_i(\theta,\varphi)$.

The currents J_j form a current matrix vector $\vec{J}$.

Allows plasma response to be viewed as a set of electrical circuits.

Singular Value Decomposition (SVD)

Suppose $\vec{\Phi} = \overleftrightarrow{M} \cdot \vec{J}$. Singular value decomposition of $\overleftrightarrow{M} \equiv \overleftrightarrow{U} \cdot \overleftrightarrow{m} \cdot \overleftrightarrow{V}^{\dagger}$, $\overleftrightarrow{U} \cdot \overleftrightarrow{U}^{\dagger} = \overleftrightarrow{1}$, $\overleftrightarrow{V} \cdot \overleftrightarrow{V}^{\dagger} = \overleftrightarrow{1}$, and $\left(\overleftrightarrow{m}\right)_{ij} = m_i \delta_{ij}$. The m_i called *singular values*.

$\left(\overleftrightarrow{A} \cdot \overleftrightarrow{B}\right)^{\dagger} = \overleftrightarrow{B}^{\dagger} \cdot \overleftrightarrow{A}^{\dagger}$, so $\overleftrightarrow{M} \cdot \overleftrightarrow{M}^{\dagger} = \overleftrightarrow{U} \cdot \left(\overleftrightarrow{m} \cdot \overleftrightarrow{m}^{\dagger}\right) \cdot \overleftrightarrow{U}^{\dagger}$, and $\overleftrightarrow{M}^{\dagger} \cdot \overleftrightarrow{M} = \overleftrightarrow{V} \cdot \left(\overleftrightarrow{m}^{\dagger} \cdot \overleftrightarrow{m}\right) \cdot \overleftrightarrow{V}^{\dagger}$

$$\overleftrightarrow{M}^{-1} \equiv \overleftrightarrow{V} \cdot \overleftrightarrow{m}^{-1} \cdot \overleftrightarrow{U}^{\dagger} \quad \text{where } \left(\overleftrightarrow{m}^{-1}\right)_{ij} = \frac{\delta_{ij}}{m_i}.$$

Let $\quad \kappa = \sum J_i f_i(\theta,\varphi) = \vec{J}^{\dagger} \cdot \vec{f}(\theta,\varphi) \quad$ and $\quad \vec{\Im} \equiv \overleftrightarrow{V}^{\dagger} \cdot \vec{J}$, so $\vec{J}^{\dagger} = \vec{\Im}^{\dagger} \cdot \overleftrightarrow{V}^{\dagger}$,

then $\quad \kappa = \vec{\Im}^{\dagger} \cdot \overleftrightarrow{V}^{\dagger} \cdot \vec{f}(\theta,\varphi) = \vec{\Im}^{\dagger} \cdot \vec{F}(\theta,\varphi)$, where $\vec{F}(\theta,\varphi) \equiv \overleftrightarrow{V}^{\dagger} \cdot \vec{f}(\theta,\varphi)$.

$$\text{If } \vec{\Phi} = \overleftrightarrow{M} \cdot \vec{J}, \text{ then} \quad \mathcal{F}_i = m_i \Im_i, \quad \text{where } \vec{\mathcal{F}} \equiv \overleftrightarrow{U}^{\dagger} \cdot \vec{\Phi}.$$

Condition number is $(m_i)_{\text{largest}} / (m_i)_{\text{smallest}}$.

6. IMPORTANT MACHINE ERRORS

The same constraints that prevent correction coils from producing pure distributions with narrow single-sign lobes also prevent machine errors from producing them.

If the main equilibrium coils produce normal fields on a control surface just on the plasma side of the coils $\delta\vec{B}_m \cdot \hat{n} = w \sum \delta\Phi_i^{(m)} f_i(\theta_c, \varphi)$.

Externally produced field on plasma boundary $\vec{\Phi}_x = \overleftrightarrow{T} \cdot \delta\vec{\Phi}_m + \overleftrightarrow{M} \cdot \vec{J}$.

$\overleftrightarrow{T}$ transfer matrix and $\overleftrightarrow{M}$ mutual inductance given by Biot-Savart.

Neil Pomphrey has developed the required code.

Plasma sensitive to I components of $\vec{\Phi}_x$. Control currents $\vec{J}$ can null I_n.

Currents in control coils should be $\vec{J} = \overleftrightarrow{M}_n^{-1} \cdot \overleftrightarrow{T}_n \cdot \delta\vec{\Phi}_m$.

$\overleftrightarrow{M}_n$ and $\overleftrightarrow{T}_n$ means restricted to the I_n components of $\vec{\Phi}_x$.

When control coil current $\vec{J} = \overleftrightarrow{M}_n^{-1} \cdot \overleftrightarrow{T}_n \cdot \delta\vec{\Phi}_m$, then the I important external distributions obey $\vec{\Phi}_x = \overleftrightarrow{\Im}_I \cdot \delta\vec{\Phi}_m$, where $\overleftrightarrow{\Im}_I \equiv \overleftrightarrow{T} - \overleftrightarrow{M} \cdot (\overleftrightarrow{M}_n^{-1} \cdot \overleftrightarrow{T}_n)$.

The locations of the control coils should be chosen to make the I-I_n non-zero singular values of $\overleftrightarrow{\Im}_I$ as small as possible.

The worst machine inaccuracies give the $\delta\vec{\Phi}_m$ that drive large singular values of $\overleftrightarrow{\Im}_I$.

The condition number of $\overleftrightarrow{M}_n^{-1} \cdot \overleftrightarrow{T}_n$ determines the required accuracy in the error field correction currents. In principle, okay if correction coils closer to plasma than error source.

The required correction currents are proportional to the machine errors and the singular values of $\overleftrightarrow{M}_n^{-1} \cdot \overleftrightarrow{T}_n$.

Measurement of As-Built Errors

Diagnostic loops can measure inaccuracy errors, $\vec{\Phi}_d = \overleftrightarrow{T}_d \cdot \delta\vec{\Phi}_m$.

The error fields on the plasma surface are $\vec{\Phi}_x^{(e)} = \overleftrightarrow{T} \cdot \delta\vec{\Phi}_m$.

Naively, $\vec{\Phi}_x^{(e)} = \overleftrightarrow{T} \cdot \overleftrightarrow{T}_d^{-1} \cdot \vec{\Phi}_d$, but meaning of $\overleftrightarrow{T}_d^{-1}$ subtle.

Only the N_d singular values of $\overleftrightarrow{T}_d$ that have a range of values smaller than the accuracy of the flux diagnostic give useful information.

Let $\overleftrightarrow{\wp}_{N_d}$ be a matrix , so that $\overleftrightarrow{\wp}_{N_d} \cdot \delta\vec{\Phi}_m$ are the components of $\delta\vec{\Phi}_m$ determined by the N_d largest singular values of $\overleftrightarrow{T}_d$, so $\overleftrightarrow{T}_d^{-1} \rightarrow \overleftrightarrow{\wp}_{N_d} \cdot \overleftrightarrow{T}_d^{-1}$.

The unmeasured errors are $\left(\vec{\Phi}_x^{(e)}\right)_u = \overleftrightarrow{T} \cdot (1 - \overleftrightarrow{\wp}_{N_d}) \cdot \delta\vec{\Phi}_m$. Must assess if plasma is sensitive to the unmeasured errors.

7. Two-Fold Degeneracy of Distributions

External magnetic distributions can be grouped in pairs with each member having an identical effect upon a tokamak plasma.

A single magnetic distribution has a definite toroidal mode number, and is proportional to $f_c(\theta,\varphi) = C(\theta)\cos N\varphi + S(\theta)\sin N\varphi$.

Due to tokamak symmetry, plasma response must be independent of toroidal phase φ_p. Same response for

$$f_c(\theta,\varphi-\varphi_p) = f_c(\theta,\varphi)\cos N\varphi_p + f_s(\theta,\varphi)\sin N\varphi_p,$$

where $f_s(\theta,\varphi) = C(\theta)\sin N\varphi - S(\theta)\cos N\varphi$.

Called cosine and sine modes because $\partial f_c/\partial\varphi = -Nf_s$ and $\partial f_s/\partial\varphi = Nf_c$.

Rotating mode means $\varphi_p = \omega t$.

Coils separated toroidally by $\pi/2N$ can specify $C(\theta)$ and $S(\theta)$.

8. Plasma Response (Global and Local)

Distortions to the plasma shape distort the equilibrium currents and can greatly amplify the perturbing magnetic field.

Plasmas can also shield perturbations:

1. Shape distortions can be amplifying or shielding.

2. If magnetic surfaces are preserved, shielding currents arise on every resonant rational surface. *Toroidal torque between plasma and the perturbation can maintain shielding currents.*

3. A Maxwell limit exists on the toroidal torque that can be exerted on a plasma by an external perturbation. *The plasma must shield out perturbations that would otherwise exceed this limit.*

Global Plasma Response

The global plasma response determines the magnetic field outside of the plasma produced by the perturbed plasma currents, which is equivalent to a surface current $\vec{J}_p$ on the plasma boundary,

$$\vec{J}_p = \vec{\rho} \cdot \vec{\Phi}_x$$

Reluctance matrix $\vec{\rho}$ completely characterizes global plasma response.

Relations involving $\vec{\rho}$ and $\overleftrightarrow{L}_p$ give Maxwellian constraints.

The flux produced by a surface current at the location of the plasma boundary is $\vec{\Phi} = \overleftrightarrow{L}_p \cdot \vec{J}$, so $\overleftrightarrow{L}_p$ is purely geometric.

In ideal MHD, $\vec{\rho}$ is Hermitian and $\delta W = \frac{1}{2}\vec{\Phi}^\dagger \cdot \overleftrightarrow{\Lambda}^{-1} \cdot \vec{\Phi}$ where $\overleftrightarrow{\Lambda} = \overleftrightarrow{L}_p + \overleftrightarrow{L}_p \cdot \vec{\rho} \cdot \overleftrightarrow{L}_p$. Note $\vec{\Phi}$ represents $\delta\vec{B} \cdot \hat{n} = \vec{B} \cdot \vec{\nabla}\vec{\xi} \cdot \hat{n}$.

Normal field on plasma boundary $\vec{\Phi} = \vec{\Phi}_x + \vec{\Phi}_p = \overleftrightarrow{A} \cdot \vec{\Phi}_x$.

$\overleftrightarrow{A} = \overleftrightarrow{1} + \overleftrightarrow{L}_p \cdot \vec{\rho}$ gives plasma amplification.

Stability s and torque α parameters defined for a given $\vec{\Phi}$:

$$\tfrac{1}{2}\vec{\Phi}^{\dagger}\cdot\overleftrightarrow{\Lambda}^{-1}\cdot\vec{\Phi} = -(s+i\alpha)\tfrac{1}{2}\vec{\Phi}^{\dagger}\cdot\overleftrightarrow{L}_p^{-1}\cdot\vec{\Phi}$$

In ideal MHD, $\alpha=0$ and positive s means plasma is unstable.

Toroidal torque plasma exerts on external magnetic field is

$$\tau_{\varphi} = -N\vec{\Phi}_x^{\dagger}\cdot\vec{\rho}_A\cdot\vec{\Phi}_x \text{ and } \left|\frac{\tau_{\varphi}}{N}\right| \le \left|\frac{\alpha}{s^2+\alpha^2}\right|\vec{\Phi}_x^{\dagger}\cdot L_p^{-1}\cdot\vec{\Phi}_x$$

Where $\vec{\rho} = \vec{\rho}_H + i\vec{\rho}_A$. Anit-Hermetian part $\vec{\rho}_A = 0$ if $\vec{\nabla}p = \vec{j}\times\vec{B}$.

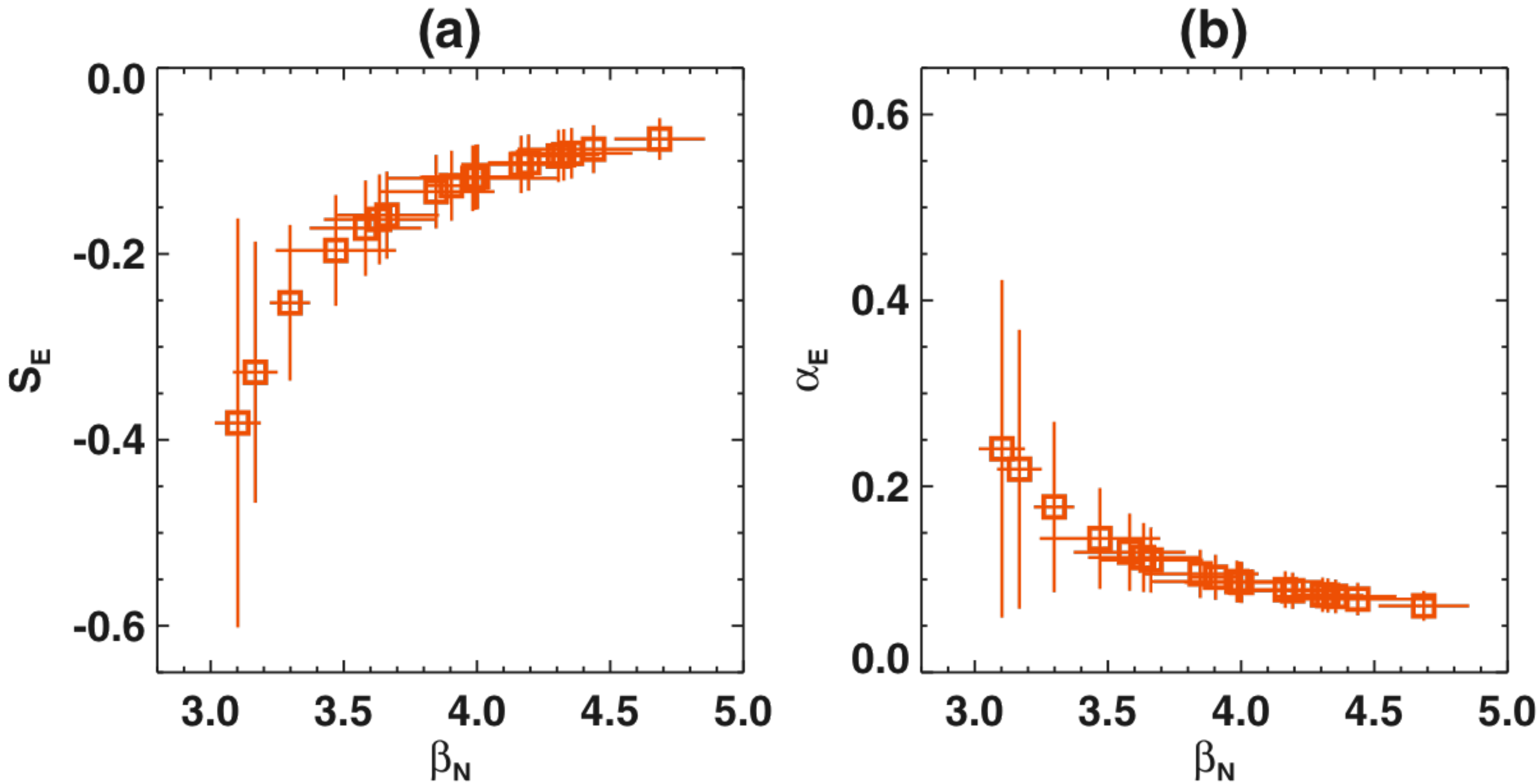
Phase shift of plasma response from external perturbation reaches π/2N when $|\alpha/s|=1$. Plasma shielding ensures $|\alpha/s|<1$.

The global response gives

- All information required for analysis of resistive wall modes.
- Much of the information required for error field control.

NSTX experiments can directly measure *s* and α and show shielding

Park *et al*, PoP **16**, 08512 (2009)



Typical α_{exp}~0.1.

Local Plasma Response

Local plasma response gives effects of magnetic perturbations on the opening of magnetic islands and on neoclassical transport.

Both have implications for the toroidal torque.

If an island opens, the plasma rotation locks to the island. Island cannot open if that exerts too large a torque.

Shielded islands exert a torque to maintain shielding currents but Maxwell's equations then imply $\alpha < \sqrt{k\rho_s}$, where $k \sim m/a$ is wavenumber of perturbation, and ρ_s width of shielding current layer.

The dominant neoclassical transport effect of asymmetry is toroidal rotation damping. A non-axisymmetric magnetic field gives a radial current depending on E_r. Can give a large α, but plasma current associated with the torque shields the perturbation.

ERROR FIELD CONTROL

Traditional analyses have studied the effect on the plasma of various Monte Carlo realizations of coil displacements.

Presupposes that the plasma effect of an arbitrary external perturbation can be assessed.

If true: the required construction accuracy and the effectiveness of the control system can be better determined by finding the external field distributions of high plasma sensitivity.

If false: error field analyses should be restricted to what is known, which is the form and the critical amplitude of the error with the highest plasma sensitivity. Sets $\delta B / B < 10^{-4}$.

The adequacy of ITER error field control coils cannot be assessed without some knowledge of the external magnetic distributions of secondary plasma sensitivity.

Error field control coils planned for ITER can balance dominate error field but are not well matched. Use of all control coils in ITER would allow much better error field control.

If information existed on external magnetic field distributions of secondary plasma sensitivity:

1. Adequacy of the error field control on ITER could be assessed.
2. Construction tolerances could be specified by spatial region.
3. Adequacy of field error measurement plan could be assessed.

Engineering tradeoff that exists between the complexities of accurate machine construction and of control coils has had little investigation.

$\delta B / B < 10^{-4}$ *probably expensive for ITER in cost and schedule.*

RESISTIVE WALL MODE CONTROL

Needed for tokamaks to achieve the level of bootstap current required for steady-state fusion systems

Conducting structures and coils, collectively called the walls, around the plasma form additional circuits with a flux $\vec{\Phi}_w$ and current $\vec{I}_w$.

Evolution of wall flux is $\dfrac{d\vec{\Phi}_w}{dt} = -\overleftrightarrow{R}_w \cdot \vec{I}_w + \vec{V}_f$

$\vec{V}_f$ is voltage applied to external circuits to provide feedback.

$$\vec{\Phi}_w = \overleftrightarrow{L}_W \cdot \vec{I}_W + \overleftrightarrow{M}_{wp} \cdot \vec{J}_p \text{ where } \vec{J}_p = \vec{\rho} \cdot \vec{\Phi}_x \text{ and } \vec{\Phi}_x = \overleftrightarrow{M}_{pw} \cdot \vec{I}_W.$$

Together with feedback logic, provide a complete set of equations for resistive wall modes.

Near marginal stability an eigenvalue of $\vec{\rho}$ scales as 1/s.

Beneficial Perturbations

ELM control best-known example

Assessment of beneficial effects generally depends on the local plasma response.

Are determined by a specific $\vec{\Phi}_x$ on plasma surface, which may require nearby coils.

While imposing desirable features, negative effect of non-axisymmetric fields must be minimized.

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